G. Dirac's (1925) formulation of QM based on commutators

Dirac (Nov 1925), *Proc. Roy. Soc. A 109 (1926) 642*

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The Fundamental Equations of Quantum Mechanics.

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In a recent paper* Heisenberg puts forward a new theory, which suggests that it is not the equations of classical mechanics that are in any way at fault, but that the mathematical operations by which physical results are deduced from them require modification. *All* the information supplied by the classical theory can thus be made use of in the new theory.

1st paragraph: About Heisenberg's theory that uses matrices

Dirac went on to (classical) Hamilton's mechanics, especially on the Poisson bracket

the Poisson (or Jacobi) bracket expression

$$[x, y] = \Sigma_r \left\{\frac{\partial x}{\partial w_r}\frac{\partial y}{\partial J_r} - \frac{\partial y}{\partial w_r}\frac{\partial x}{\partial J_r}\right\} = \Sigma_r \left\{\frac{\partial x}{\partial q_r}\frac{\partial y}{\partial p_r} - \frac{\partial y}{\partial q_r}\frac{\partial x}{\partial p_r}\right\}$$

Hamilton's mechanics

We make the fundamental assumption that *the difference between the Heisenberg products of two quantum quantities is equal to $ih/2\pi$ times their Poisson bracket expression.* In symbols,

$$xy - yx = ih/2\pi \,.\, [x, y]. \qquad (11)$$

Dirac 1925

coordinate $x \rightarrow \hat{x}$

momentum $p \rightarrow \hat{p}$

Classical Mechanics gives $\underbrace{\{x, p\}_{PB}}_{\text{Poisson Bracket}} = 1$

Dirac says: In Quantum Mechanics, impose

$$\underbrace{[\hat{x}, \hat{p}]}_{\hat{x}\hat{p} - \hat{p}\hat{x}} = i\hbar \{x, p\}_{PB} = i\hbar$$

then everything follows (doesn't matter how it is imposed)

(Key point: In Dirac's view, there is **all** of QM. And it is true! It is a fundamental property of Nature, described mathematically!)

Dirac's method of "canonical quantization" provides a systematic way to do quantum mechanics, once *L* and *H* of a system are known! Part of his 1933 Nobel Prize.

Revisiting Schrodinger's Quantum Mechanics (1926) from Dirac's viewpoint

Harmonic Oscillator

momentum conjugated to coordinate

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

coordinate-momentum pair

coordinate

Go Quantum!

$$\hat{H}\psi = E\psi$$

Schrodinger Equation (1926) was built on the Hamiltonian

coordinate → coordinate operator $\hat{x}$ $\quad \hat{x} \to x$

momentum → momentum operator $\hat{p}$ $\quad \hat{p} \to -i\hbar\frac{\partial}{\partial x}$

coordinate [whatever it is]

[This is ***a way*** to meet Dirac's assumption]

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2\right]\psi(x) = E\psi(x)$$

$$\therefore \boxed{\left[x, \frac{\hbar}{i}\frac{d}{dx}\right] = i\hbar}$$

[this is part of Dirac's 1933 Nobel Prize for his 1925 work]

The point is: Schrodinger Equation turned out to be ***a way*** to satisfy Dirac's assumption on the commutator [x,p]. Of course, Schrodinger did not think about his equation this way back in 1925.

We will stick to Schrodinger's way of doing QM for a large part of our course.

Names related to Schrodinger's way of doing QM:

- Schrodinger picture [it is the wavefunction that evolves in time, i.e., $\Psi(x,t)$ as we have been discussing so far]
- Schrodinger representation

Any other way to satisfy Dirac's $[\hat{x}, \hat{p}] = i\hbar$ assumption?

[Note: The discussion here is meant to bring out the point that Schrodinger's way is not the only way to do QM.]

Nothing special about the position (x) space, except that our sensation of position comes naturally

How about $\hat{x} \to i\hbar \frac{\partial}{\partial p}$ and $\hat{p} \to p$?

$$[\hat{x}, \hat{p}]\, \mathcal{F}(p) = i\hbar \frac{\partial}{\partial p}\left(p\, \mathcal{F}(p)\right) - p\, i\hbar \frac{\partial}{\partial p} \mathcal{F}(p)$$

$$= i\hbar\, \mathcal{F}(p) \quad \text{for all } \mathcal{F}(p)$$

[This would have been the way to develop TISE if our eyes were more sensitive to momentum (when we see someone, we already evaluate mass x velocity in our mind) than to position.]

Harmonic Oscillator in p-space (momentum space)

$$\frac{\hat{p}^2}{2m}\mathcal{F}(p) + \frac{1}{2}K\left(i\hbar\frac{\partial}{\partial p}\right)\left(i\hbar\frac{\partial}{\partial p}\right)\mathcal{F}(p) = E\,\mathcal{F}(p)$$

or $$\frac{p^2}{2m}\mathcal{F}(p) - \frac{\hbar^2}{2}K\frac{\partial^2}{\partial p^2}\mathcal{F}(p) = E\,\mathcal{F}(p)$$

This is the QM "Schrödinger" Equation in momentum space

How about $\psi_{\text{oscillator}}(x)$? By Fourier Transform

Stop here! This is as much as momentum-space approach as we will do in this course.

Any other way? How about Matrices?

Our strategy: Let's get used to doing QM in Schrodinger Equations. It is the easiest approach for development our quantum sense. Only after that, you could play with different ways of doing QM.

Optional Part: Quantizing EM fields – Let's put the ideas together

Dirac's way of imposing a commutator to "go quantum" is called "**canonical quantization**". It provides a systematic way to do quantum mechanics, once *L* and *H* of a system are known!

The method has far reaching consequences. It is being used in

- Dirac's equation of an electron (relativistic QM)
- Operator methods in solving QM problems
- Quantum many-body theories
- Quantizing classical fields (e.g. quantizing EM fields -> what are photons?)
- Quantizing already quantized equations => quantum field theories QFT (QED, QCD, electroweak theory)
- See beginning chapter of any book on QFT

Optional Part: Quantizing EM fields without Mathematics

Think like a physicist!

Flip the arguments and see if it works!

Do we know the equation of motion?

Yes, we have Maxwell's equations (fortunately)

1. Flip the argument – Find *Lagrangian* that gives the right (known) equation when it is plugged into Euler-Lagrange equation

2. Then we know the coordinates (e.g. vector field A) and their corresponding momenta

3. From *L* get *H* for EM fields in terms of coordinates and momenta

4. Then impose Dirac's canonical quantization $\underbrace{[\hat{x}, \hat{p}]}_{\hat{x}\hat{p} - \hat{p}\hat{x}} = i\hbar\{x, p\}_{PB} = i\hbar$

[EM fields are "quantized" in Step 4]

For coordinate-momentum pairs

The resulting EM Hamiltonian is found to be a set of quantum harmonic oscillators!

From QM oscillator physics

$$E_n = (n + \tfrac{1}{2})\hbar\omega \qquad n = 0,1,2,3 \ldots$$

$$\omega = 2\pi(\text{frequency})$$

$$\hbar = \text{Planck's constant}/2\pi$$

- Ground state has some (zero point) energy but n=0 excitations
- First excited state has n=1 quanta of energy above ground state, etc.

- Thus, we have photons!
- Ground state is vacuum (but with much energy)
- Excited states have n=1,2,… photons (quanta of energy)

We have just done the beginning of Quantum Field Theory by (words/conceptually) following the ideas developed in this Chapter without mathematics!

[This ends the optional part]